"It's really quite straightforward, there's nothing mystical about it."

- Prof. Eugene D. Commins, U.C. Berkeley, on the subject of the Einstein-Podolsky-Rosen paradox.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

The average energy of a particle $\bar{E}$ in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$ is given by:

$$\bar{E} = \frac{U}{N}, \tag{1}$$

where U is the total energy of N particles in identical potentials (this is the usual imaginary ensemble of identical quantum mechanical systems used to calculate expectation values, etc.). The total energy is given by:

$$U = \sum_{n=0}^{\infty} N_0 E_n e^{-E_n \beta} = \sum_{n=0}^{\infty} N_0 \hbar\omega_0 \left(n + \frac{1}{2}\right) e^{-n\beta\hbar\omega_0}, \tag{2}$$

where $\beta = 1/(kT)$ and the total number of particles is:

$$N = \sum_{n=0}^{\infty} N_0 e^{-n\beta\hbar\omega_0}. \tag{3}$$

The average energy is then given by the expression:

$$\frac{U}{N} = \frac{\sum_{n=0}^{\infty} N_0 \hbar\omega_0 n e^{-n\beta\hbar\omega_0}}{\sum_{n=0}^{\infty} N_0 e^{-n\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0. \tag{4}$$

Cancelling out common factors of N_0 and noting that

$$(n\hbar\omega_0)e^{-\beta n\hbar\omega_0} = -\frac{\partial}{\partial\beta}e^{-\beta n\hbar\omega_0},$$

we can simplify Eq. (4) to:

$$\frac{U}{N} = \frac{-\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0}}{\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0. \tag{5}$$

Next we use the fact that for a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Thus for the average energy, we have:

$$\bar{E} = \frac{-\frac{\partial}{\partial\beta}\left(\frac{1}{1-e^{-\beta\hbar\omega_0}}\right)}{\frac{1}{1-e^{-\beta\hbar\omega_0}}} + \frac{1}{2}\hbar\omega_0. \tag{6}$$

From which we find:

$$\bar{E} = \frac{\hbar\omega_0 e^{-\beta\hbar\omega_0}}{1-e^{-\beta\hbar\omega_0}} + \frac{1}{2}\hbar\omega_0 = \frac{\hbar\omega_0}{e^{\hbar\omega_0/kT}-1} + \frac{1}{2}\hbar\omega_0. \tag{7}$$

This is an expression for the average energy of the particle at any temperature. If we take the high temperature limit ($kT \gg \hbar\omega_0$), $e^{\beta\hbar\omega_0} \approx 1 + \beta\hbar\omega_0$, so

$$\boxed{\bar{E} \approx \frac{1}{\beta} = kT.}$$

Problem 2

Here, we consider the potential $\frac{1}{2}m\omega^2 x^2$ of the harmonic oscillator, where ω is a constant. We define the operator $A \equiv y + iq$, where $y \equiv x\sqrt{m\omega/2}$ and $q \equiv p/\sqrt{2m\omega}$. Since $\langle x\rangle$ and $\langle p\rangle$ are physical observables, $x^\dagger = x$ and $p^\dagger = p$ and subsequently $y^\dagger = y$ and $q^\dagger = q$.

(a)
$$A^\dagger = (y + iq)^\dagger = y^\dagger - iq^\dagger = y - iq$$

(b)

$$[A, A^\dagger] = (y + iq)(y - iq) - (y - iq)(y + iq) = [y, y] + [y, -iq] + [iq, y] + [iq, -iq]$$

We know that y and q commute with themselves, and that

$$[y, q] = -[q, y].$$

Employing these results, we find that

$$[A, A^\dagger] = -2i[y, q].$$

From the definition of y and q,

$$[y, q] = \left[x\sqrt{\frac{m\omega}{2}}, \frac{p}{\sqrt{2m\omega}} \right] = \frac{1}{2}[x, p] = i\hbar/2.$$

From which we conclude:

$$\left[A, A^\dagger\right] = -2i[y, q] = \hbar.$$

(c)

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2$$

Since $p^2 = 2m\omega q^2$ and $x^2 = 2y^2/(m\omega)$, we find

$$\mathcal{H} = \omega(q^2 + y^2).$$

(d),(e)   Note that from the definitions of A and $A^\dagger$, we have

$$y = \frac{A + A^\dagger}{2}$$

and

$$q = \frac{A - A^\dagger}{2i}.$$

Thus we find that

$$y^2 = \frac{1}{4}\left(A^2 + A^{\dagger 2} + AA^\dagger + A^\dagger A\right)$$

and

$$q^2 = \frac{1}{4}\left(-A^2 - A^{\dagger 2} + AA^\dagger + A^\dagger A\right)$$

From (c) and the above considerations we have that

$$\mathcal{H} = \omega(q^2 + y^2) = \frac{\omega}{2}\left(AA^\dagger + A^\dagger A\right).$$

By adding and subtracting $AA^\dagger$ or $A^\dagger A$ where appropriate,

$$AA^\dagger = AA^\dagger - A^\dagger A + A^\dagger A = \left[A, A^\dagger\right] + A^\dagger A = \hbar + A^\dagger A$$

and also

$$A^\dagger A = A^\dagger A - AA^\dagger + AA^\dagger = \left[A^\dagger, A\right] + AA^\dagger = -\hbar + AA^\dagger.$$

These expressions can be used in our above expression for $\mathcal{H}$, and from them we find

$$\mathcal{H} = \frac{\omega}{2}\left(AA^\dagger + A^\dagger A\right) = \omega\left(A^\dagger A + \hbar/2\right) = \omega\left(AA^\dagger - \hbar/2\right).$$

(f) Since constants commute with anything $\left[\hbar/2, A^\dagger\right] = 0$. Thus we get:

$$\left[\mathcal{H}, A^\dagger\right] = \left[\omega AA^\dagger, A^\dagger\right] = \omega\left(AA^\dagger A^\dagger - A^\dagger AA^\dagger\right) = \omega\left[A, A^\dagger\right]A^\dagger = \hbar\omega A^\dagger.$$

(g) Similarly,

$$\left[\mathcal{H}, A\right] = \left[\omega AA^\dagger, A\right] = \omega\left(AA^\dagger A - AAA^\dagger\right) = \omega A\left[A^\dagger, A\right] = -\hbar\omega A.$$

Problem 3

Consider the angular momentum operator $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = (\hbar/i)\mathbf{r} \times \nabla$. For example, $L_x \equiv (\hbar/i)\left(y(\partial/\partial z) - z(\partial/\partial y)\right)$. Define $L_\pm \equiv L_x \pm iL_y$.

(a)

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [zp_y, xp_z] - [zp_y, zp_x] - [yp_z, xp_z] \tag{8}$$

Recall that in figuring out these commutation relations, it often helps to think of the commutators as operators acting on functions. This is especially helpful in dealing with commutators involving derivatives. We'll look at each of the terms in the above expression individually:

$$[yp_z, zp_x] = -\hbar^2 y\frac{\partial}{\partial z}z\frac{\partial}{\partial x} + \hbar^2 z\frac{\partial}{\partial x}y\frac{\partial}{\partial z}$$

$$[yp_z, zp_x] = -\hbar^2 yz\frac{\partial^2}{\partial z\partial x} - \hbar^2 y\frac{\partial}{\partial x} + \hbar^2 yz\frac{\partial^2}{\partial x\partial z} = -\hbar^2 y\frac{\partial}{\partial x}$$

$$\boxed{[yp_z, zp_x] = -\hbar^2 y\frac{\partial}{\partial x}}$$

$$[zp_y, xp_z] = -\hbar^2 z\frac{\partial}{\partial y}x\frac{\partial}{\partial z} + \hbar^2 x\frac{\partial}{\partial z}z\frac{\partial}{\partial y}$$

$$[zp_y, xp_z] = -\hbar^2 zx\frac{\partial^2}{\partial y\partial z} + \hbar^2 x\frac{\partial}{\partial y} + \hbar^2 xz\frac{\partial^2}{\partial z\partial y}$$

$$\boxed{[zp_y, xp_z] = \hbar^2 x\frac{\partial}{\partial y}}$$

$$[zp_y, zp_x] = -\hbar^2 z\frac{\partial}{\partial y}z\frac{\partial}{\partial x} + \hbar^2 z\frac{\partial}{\partial x}z\frac{\partial}{\partial y}$$

$$[zp_y, zp_x] = -\hbar^2 z^2\frac{\partial^2}{\partial y\partial x} + \hbar^2 z^2\frac{\partial^2}{\partial x\partial y}$$

$$\boxed{[zp_y, zp_x] = 0}$$

$$[yp_z, xp_z] = -\hbar^2 y \frac{\partial}{\partial z} x \frac{\partial}{\partial z} + \hbar^2 x \frac{\partial}{\partial z} y \frac{\partial}{\partial z}$$

$$[yp_z, xp_z] = -\hbar^2 yx \frac{\partial^2}{\partial z^2} + \hbar^2 xy \frac{\partial^2}{\partial z^2}$$

$$\boxed{[yp_z, xp_z] = 0}$$

Now we can put these simplified expressions into Eq. (8), and we find:

$$[L_x, L_y] = \hbar^2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = i\hbar \left(x \left(\frac{\hbar}{i} \frac{\partial}{\partial y} \right) - y \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \right)$$

$$[L_x, L_y] = i\hbar (x p_y - y p_x)$$

$$\boxed{[L_x, L_y] = i\hbar L_z}$$

(b),(c) The arguments used in (a) can be basically repeated, just changing the identities of some of the variables. Or you can argue that since space is rotationally invariant, if we rotate our coordinate system in such a way that $x \to y$, $y \to z$ and $z \to x$, the same commutation relation holds with the appropriate change of coordinate names. The basic principle is that for any (i, j, k) which are a cyclic permutation of (x, y, z), we have:

$$[L_i, L_j] = i\hbar L_k.$$

We know that in general if we interchange two operators in a commutator, the result of the commutator acquires a negative sign. In other words, for any two operators A, B:

$$[A, B] = -[B, A].$$

Thus if (i, j, k) are an anti-cyclic permutation of (x, y, z) (e.g., (y, x, z)), we have:

$$[L_i, L_j] = -i\hbar L_k.$$

(d)

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = [L_x, L_x] + i[L_y, L_x] - i[L_x, L_y] + [L_y, L_y]$$

An operator always commutes with itself, so

$$[L_x, L_x] = 0, \quad [L_y, L_y] = 0.$$

Thus we have

$$[L_+, L_-] = -2i[L_x, L_y] = -2i(i\hbar L_z)$$

$$\boxed{[L_+, L_-] = 2\hbar L_z}$$

(e)

$$[L_-, L_z] = [L_x - iL_y, L_z] = [L_x, L_z] - i[L_y, L_z]$$

$$[L_-, L_z] = -i\hbar L_y - i(i\hbar L_x) = \hbar(L_x - iL_y)$$

$$\boxed{[L_-, L_z] = \hbar L_-}$$

(f)

$$[L_+, L_z] = [L_x + iL_y, L_z] = [L_x, L_z] + i[L_y, L_z]$$

$$[L_+, L_z] = -i\hbar L_y + i(i\hbar L_x) = -\hbar(L_x + iL_y)$$

$$\boxed{[L_+, L_z] = -\hbar L_+}$$

(g)

$$[L^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z]$$

$$[L^2, L_z] = L_x L_x L_z - L_z L_x L_x + L_y L_y L_z - L_z L_y L_y$$

Now we employ a common trick in calculating commutation relations. We add and subtract terms which allow us to substitute commutators we know into our expression.

$$[L^2, L_z] = L_x L_x L_z - L_x L_z L_x + L_x L_z L_x - L_z L_x L_x + L_y L_y L_z - L_y L_z L_y + L_y L_z L_y - L_z L_y L_y$$

$$[L^2, L_z] = L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y$$

Maybe, if you are like me, when you first see this trick it seems quite clever. This trick is used very often because of the potential non-commutativity of operators. Suppose we have two operators A, B which don't commute. If we have AB and want to get BA for some reason, we can use

$$AB = AB - BA + BA = [A, B] + BA.$$

This is a very useful operator identity! Anyhow, continuing on with the math by substituting in results from (a),(b) and (c) of this problem:

$$[L^2, L_z] = -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y$$

$$\boxed{[L^2, L_z] = 0}$$

This relation proves it is possible to find simultaneous eigenfunctions of both operators. This is an important result, so it probably won't hurt to see why this is the case once more. Suppose we have operators A, B where $[A, B] = 0$. Consider an eigenfunction of B, ψ_b, with eigenvalue λ_b. Is $A\psi_b$ an eigenfunction of B? It is, since

$$BA\psi_b = AB\psi_b = A(\lambda_b \psi_b) = \lambda_b(A\psi_b).$$

As you can see, this result relies on the fact that A and B commute. Thus measurement of one observable does not affect measurement of the other observable. This shows that there is no fundamental quantum uncertainty in measurement of observables which correspond to commuting operators. So we can find a function $\psi_a b$ which is is an eigenfunction of both A and B with eigenvalues λ_a and λ_b, as we do for L^2 and L_z.

(h) We can use arguments analagous to those above to show that:

$$\left[L^2, L_x\right] = 0, \quad \left[L^2, L_y\right] = 0.$$

If you want to avoid the math, you can just use the isotropy of space to claim that L^2 should not preferentially commute with a particular direction in space. It immediately follows that

$$\boxed{\left[L^2, L_\pm\right] = 0}$$

(i),(j) First, let's consider L_+L_- and L_-L_+:

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + iL_yL_x - iL_xL_y + L_y^2$$

$$L_+L_- = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$$

To go from L_+L_- to L_-L_+ we use the identity

$$L_+L_- = [L_+, L_-] + L_-L_+ = L_-L_+ + 2\hbar L_z.$$

Since

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

and

$$L_+L_- = L_x^2 + L_y^2 + \hbar L_z, \quad L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$$

we have:

$$\boxed{L^2 = L_-L_+ + L_z^2 + \hbar L_z}$$

$$\boxed{L^2 = L_+L_- + L_z^2 - \hbar L_z}$$

Problem 4

The spherical harmonic $Y_{lm}(\theta, \phi)$ is an eigenfunction of L^2 with eigenvalue $\hbar^2 l(l+1)$ and also of L_z with eigenvalue $\hbar m$. It is normalized so that

$$\int d\Omega \, Y_{lm}^*(\theta, \phi)Y_{lm}(\theta, \phi) = 1.$$

(a)

Consider the integral:

$$\int d\Omega (L_+Y_{ll})^*(L_+Y_{ll}) = \int d\Omega Y_{ll}^*(L_-L_+)Y_{ll}$$

where we make use of the fact that:

$$(L_+Y_{ll})^* = Y_{ll}^*L_+^\dagger = Y_{ll}^*L_-$$

From problem 3 parts (i),(j) we have:

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z.$$

Conveniently, Y_{ll} is an eigenfunction of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar l$ respectively. Thus we obtain

$$\int d\Omega (L_+Y_{ll})^*(L_+Y_{ll}) = -\int d\Omega Y_{ll}^*\hbar^2\big(l(l+1) - l^2 - l\big)Y_{ll} = 0.$$

Therefore, it must be that

$$\boxed{L_+Y_{ll} = 0}$$

(b)

We can use quite similar methods to find $C_-(l, m)$. Consider the integral:

$$\int d\Omega (L_-Y_{lm})^*(L_-Y_{lm}) = \int d\Omega Y_{lm}^*(L_+L_-)Y_{lm}$$

From problem 3 parts (i),(j) we have:

$$L_+L_- = L^2 - L_z^2 + \hbar L_z.$$

Y_{lm} is an eigenfunction of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$ respectively. Thus we have the integral

$$\int d\Omega (L_-Y_{lm})^*(L_-Y_{lm}) = \int d\Omega Y_{lm}^*\hbar^2\big(l(l+1) - m^2 + m\big)Y_{lm}$$

Because the Y_{lm}'s are orthonormal, we have that

$$|C_-(l,m)|^2 = \hbar^2 \left(l(l+1) - m^2 + m \right)$$

n_1	n_2	n_3	Energy
1	1	1	6
1	2	1	9
2	1	1	9
2	2	1	12
1	3	1	14
3	1	1	14
2	3	1	17
3	2	1	17

Problem 5

(a)

Since the atom is in a state of definite projection $m_j = 5/2$ of its *total* (spin + orbital) angular momentum on the z axis, $j \geq 5/2$. This follows from the fact that possible values for m_j range between $+j$ and $-j$. What values of j are possible for a one electron atom with orbital angular momentum $l = 2$? The values of j range between $l + s$ and $|l - s|$ where s is the electron spin. Therefore, in general $j = 5/2, 3/2$ are possible, but since the atom is in the $m_j = 5/2$ state we know $j = 5/2$.

(b)

Consider

$$J^2 = (\mathbf{L} + \mathbf{S})^2 = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}.$$

Since the atom is in an eigenstate of J^2, L^2 and S^2, we have that:

$$\mathbf{L} \cdot \mathbf{S} = \frac{\hbar^2}{2}(j(j+1) - l(l+1) - s(s+1)).$$

With $j = 5/2$, $l = 2$, and $s = 1/2$, we obtain $\mathbf{L} \cdot \mathbf{S} = \hbar^2$. Also,

$$\sqrt{\langle L^2 \rangle \langle S^2 \rangle} = \hbar^2 \sqrt{l(l+1) \cdot s(s+1)} = \hbar^2 \frac{3}{\sqrt{2}}.$$

So we find,

$$\boxed{\langle \cos\theta \rangle = \frac{\sqrt{2}}{3}}$$

Problem 6   *Rohlf 7.27*

The energies for the particle in the 3D box are given by:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2 \pi^2}{2mL^2}(n_1^2 + n_2^2 + 4n_3^2).$$

The following table shows the first 5 unique energies (in units of $\frac{\hbar^2 \pi^2}{2mL^2}$) and the quantum numbers of the states that possess them.

Problem 7   *Rohlf 8.21*

(a)

Of course, first we specify that we know l and s. The possible values for j are $j = l + s, l + s - 1, ..., |l - s|$. We choose one particular value of j. Then there are $2j + 1$ states with total angular momentum j. There are many ways to show this result. You could start with a "stretched" state ($m_j = \pm j$) and use the raising or lowering operator, for example. But if you simply note that m_j can take on the possible values

$$m_j = j, j - 1, ..., -j + 1, -j$$

and count these states, we see immediately that the number of states is $2j + 1$.

(b)

The quantum numbers of the $n = 2$ states of hydrogen in terms n, l, m_l and m_s are shown in the following table:

n	l	m_l	m_s
2	1	1	1/2
2	1	1	-1/2
2	1	0	1/2
2	1	0	-1/2
2	1	-1	1/2
2	1	-1	-1/2
2	0	0	1/2
2	0	0	-1/2

Note there are 8 states in total in the table. The quantum numbers of the $n = 2$ states of hydrogen in terms n, l, j and m_j are shown in the following table:

n	l	j	m_j
2	1	3/2	3/2
2	1	3/2	1/2
2	1	3/2	-1/2
2	1	3/2	-3/2
2	1	1/2	1/2
2	1	1/2	-1/2
2	0	1/2	1/2
2	0	1/2	-1/2

Here there are also 8 states. The system we are considering is described by an 8D Hilbert space, so any complete, orthonormal set of eigenfunctions which span the space must consist of 8 states.

(c)

Since $J_z = L_z + S_z$, for the state $m_j = 3/2$, $l = 1$, $m_l = 1$ and $m_s = 1/2$. This is the "stretched" state, and we can readily convert from the n, l, j and m_j basis to the n, l, m_l and m_s basis by making the correspondence between the stretched states and employing the raising and lowering operators. Note also that $J_\pm = L_\pm + S_\pm$. Such transformations are used quite often and are tabulated (these are the famed Clebsch-Gordan coefficients).

(d)

If $m_j = 1/2$, then we can have

$$(l, m_l, m_s) = (0, 0, 1/2)$$

$$(l, m_l, m_s) = (1, 0, 1/2)$$

$$(l, m_l, m_s) = (1, 1, -1/2).$$

Problem 8   *Rohlf 8.25*

(a)

The magnetic dipole moment $\vec{\mu}$ of a hydrogen atom, in the limit of a strong $\vec{B}$-field, is given by:

$$\vec{\mu} = -\frac{e}{2m}(\mathbf{L} + 2\mathbf{S})$$

and the energy shift ΔE due to the external field is given by

$$\Delta E = -\vec{\mu} \cdot \vec{B}.$$

So we get energy shifts proportional to $m_l + 2m_s$.

For hydrogen in the n=3 state with a strong $\vec{B}$-field, we have the following possible values for the angular momentum quantum numbers and energy shifts in units of $e\hbar B/(2m)$:

m_l	m_s	ΔE
2	1/2	3
2	-1/2	1
1	1/2	2
1	-1/2	0
0	1/2	1
0	-1/2	-1
-1	1/2	0
-1	-1/2	-2
-2	1/2	-1
-2	-1/2	-3

(b)

In the absence of a magnetic field the energy separation ΔE_0 of the $3p$ and $1s$ states is

$$\Delta E_0 = 13.6 \text{ eV} - \frac{13.6}{9} \text{ eV} = 12.1 \text{ eV}.$$

The electric dipole transition selection rules demand that the difference in the projection of the orbital angular momentum on the z-axis between the initial and final states of an atomic transition must obey

$$\Delta m_l = 1, 0, -1.$$

So the energies of the photons E_γ can be

$$E_\gamma = 12.1 \text{ eV} \pm \frac{e\hbar B}{2m}, \; 12.1 \text{ eV}.$$